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GRAVITY INDUCED POSITION ERRORS IN AIRBORNE INERTIAL NAVIGATION

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The report investigates the feasibility of improving airborne inertial navigation by use of gravity field approximations which are more accurate than the normal model presently applied. The effect of the anomalous gravity field on positioning is investigated by using a simplified dynamical error model and by deriving analytical expressions for the steady state error via the state space approach. In this approach, changes in the anomalous gravity field are cast into the form of first-order differential equations which are related to a position dependent covariance representation of the gravity field

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) by way of the vehicle velocity. Different possibilities for a state space model of the anomalous field are discussed. The procedure chosen combines the consistency of the Tscherning-Rapp model with the advantages of a formulation in terms of Gauss-Markov processes by making use of the essential parameters of a covariance function proposed by Moritz. The expressions for the gravity induced position errors resulting from this approach are easy to compute for a wide variety of cases. The assumptions made to derive them are in general justifiable.

Sigma

Based on the available gravity field information a number of approximation models are proposed and expressed in terms of equivalent spherical harmonic expansions. Results show that the use of presently available global models would reduce the gravity induced position errors from σ = 150 m. Improved global models expected in the near future, as for instance those from the GRAVSAT mission, would bring errors below σ = 50 m. However, to reach the meter range, a gravity field approximation equivalent to an expansion of degree and order 1000 would be necessary. This result is not surprising. It demonstrates the well-known fact that the medium and high frequency spectrum contributes considerably to the deflections of the vertical or, in other words, that the relative contribution of local effects is not negligible in this case.

Considering the accuracy of present day inertial sensors, gravity field models giving σ = 150 m seem to be adequate and it may take some time before non-gravitational system errors in airborne navigation can be reduced to a level of σ = 50 m.

Sigma

FOREWORD

This report was prepared by Klaus-Peter Schwarz, Associate Professor, The University of Calgary, under Air Force Contract No. F19628-79-C-0075, The Ohio State University Research Foundation, Project No. 711715, Project Supervisor, Urho A. Uotila, Professor, Department of Geodetic Science and Surveying. The contract covering this research is administered by the Air Force Geophysics Laboratory (AFGL), Hanscom Air Force Base, Massachusetts, with George Hadgigeorge/LW, Contract Monitor.

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1. INTRODUCTION

Errors in inertial navigation can be subdivided into two groups, the system related errors, and the modelling errors. The first group comprises e.g. measuring noise, calibration and alignment errors while the second group contains approximation, linearization and lumping errors. Traditionally, advances in system technology which resulted in a reduction of the system related errors have always called for a scrutiny of the underlying models. For a long time, one of the approximations, the normal gravity model, proved to be sufficient for airborne inertial navigation. However, with sensor accuracies predicted for third generation inertial systems (Draper, 1977), this will not be true anymore. This report therefore studies the size and characteristical behaviour of gravity induced position errors in airborne inertial navigation.

In order to relate the accuracy of a given gravity field model to the system errors, the effect of model approximations on system performance must be studied. This topic is by no means a new one. A number of papers in the late sixties and early seventies dealt with it and discussed various aspects of the problem. Since they were written by systems engineers and were published in journals seldom read by geodesists, they had almost no impact on geodesy. This is regrettable for two reasons. On the one hand, the representation of a velocity transformed gravity field adds an interesting new dimension to the usual geodetic approach. On the other hand, geodetic results on the covariance structure of the gravity field could have been readily used to generate models consistent with our present knowledge. The objective of this report is therefore twofold. On the one hand, it will present the basic solution approach without assuming a background in systems theory. It is hoped that in this way some interest in this fascinating area will be stirred among geodesists. On the other hand, an attempt will be made to incorporate recent geodetic results into the solution and to point out areas where further research is needed.

2. THE DYNAMIC ERROR MODEL

Inertial navigation is based on Newton's second law. Thus, the basic mathematical model is a system of second-order differential equations. Such a system can always be transformed into a system of first-order differential equations. The derivation of the first-order model from the basic set of equations is discussed in detail in Britting (1971) for the different mechanizations used in inertial navigation. The resulting dynamic system usually is of the state space variety, i.e. it is expressed in terms of a time dependent vector x(t) whose components at any specified time represent the state of the system. The dynamic error model is a combination of a state space formulation and white noise processes. Such a model offers a considerable flexibility in applications and takes into account that the error process contains both deterministic and stochastic elements. General characteristics of the dynamic error model will be discussed in this section while the specific formulation of the inertial navigation problem will be the topic of section 5.

In its most general form the linear dynamic error model is given by

$$\dot{\underline{x}}(t) = \underline{F}(t)\underline{x}(t) + \underline{G}(t)\underline{w}(t) + \underline{L}(t)\underline{u}(t) \qquad (2.1a)$$

with initial conditions

$$\underline{x}(o) = \underline{c} \qquad (2.1b)$$

and control measurements

$$\underline{z}(t) = \underline{H}(t)\underline{x}(t) + \underline{r}(t) \qquad (2.1c)$$

where

$\underline{x}(t)$ ... is the state vector of the system
$\dot{\underline{x}}(t)$ ... is the time derivative of $\underline{x}$
$\underline{F}(t)$ ... is the dynamics matrix
$\underline{G}(t)\underline{w}(t)$ ... is the system noise (stochastic)
$\underline{L}(t)\underline{u}(t)$ ... is the control function (deterministic)
$\underline{z}(t)$ ... is the measurement

$\underline{H}(t)$ ... is the design matrix, relating measurement and state vector

$\underline{r}(t)$ ... is the measurement noise

and where all underlined quantities represent vectors (lower case) or matrices (upper case). In these equations it is assumed that the matrices $\underline{F}$, $\underline{G}$, $\underline{H}$ are completely known, i.e. the uncertainty enters via $\underline{w}(t)$, $\underline{r}(t)$, and $\underline{x}(o)$. The variables $\underline{w}(t)$ and $\underline{r}(t)$ are stochastic white noise processes which do not have a time structure. The knowledge of the process value at one instant of time does not provide knowledge about its values at any other instant. These processes are completely defined by their first and second moments

$$E\{\underline{w}(t)\} = 0 \qquad E\{\underline{w}(t_1)\underline{w}^T(t_2)\} = \delta(t_1-t_2)Q(t) \tag{2.2a}$$

$$E\{\underline{r}(t)\} = 0 \qquad E\{\underline{r}(t_1)\underline{r}^T(t_2)\} = \delta(t_1-t_2)R(t) \tag{2.2b}$$

where E{ . } denotes the statistical expectation and $\delta(t_1-t_2)$ is the Dirac delta function. This definition does not assume that the process is Gaussian because the probability densities of $\underline{w}(t)$ and $\underline{r}(t)$ may be specified in any way. The initial conditions are modelled as a random vector

$$E\{\underline{x}(o)\} = 0 \qquad E\{\underline{x}(o)x^T(o)\} = \underline{P}(o) \tag{2.3}$$

The model expressed by equations (2.1) to (2.3) has the important property that $\underline{x}(o)$, $\underline{w}(t)$, and $\underline{r}(t)$ are not related, i.e.

$$E\{\underline{x}(o)\underline{w}^T(t)\} = 0 \tag{2.4a}$$

$$E\{\underline{x}(o)\underline{r}^T(t)\} = 0 \tag{2.4b}$$

$$E\{\underline{w}(t)\underline{r}^T(t)\} = 0 \tag{2.4c}$$

The deterministic input $\underline{L}(t)\underline{u}(t)$ is usually considered as known which for a linear model like (2.1) means that its effect can be added or subtracted. It will therefore be disregarded in the sequel. A further simplification can be obtained by requiring that $\underline{F}$, $\underline{G}$, $\underline{H}$, $\underline{Q}$, and $\underline{R}$ are matrices with constant coefficients. In this case a time invariant model is obtained which is of the form

$$\underline{\dot{x}}(t) = \underline{\underline{F}}\underline{x}(t) + \underline{\underline{G}}\underline{w}(t) \tag{2.5a}$$

with $\underline{x}(o) = \underline{c}$ (2.5b)

and $\underline{z}(t) = \underline{\underline{H}}\underline{x}(t) + \underline{r}(t)$. (2.5c)

It should be noted that such a model does not imply that $\underline{x}(t)$ is a stationary stochastic process. The advantage of model (2.5) versus model (2.1) is that statements on the stability of the system are usually simpler and that it is sometimes possible to find an analytical solution to the homogeneous system of differential equations

$$\underline{\dot{x}}(t) = \underline{\underline{F}}\underline{x}(t) \ . \tag{2.6}$$

Such solutions simplify the error analysis and are usually the only way to arrive at reliable error estimates without excessive use of computer time. Equation (2.5) will therefore be considered as the basic model for all the following discussions.

The formal solution of equation (2.5) can be written as

$$\underline{x}(t) = \underline{\Phi}(t,t_0)\underline{c} + \int_{t_0}^{t} \underline{\Phi}(t,s)\underline{\underline{G}}\underline{w}(s)ds \tag{2.7}$$

where $\underline{\Phi}(t,t_0)$ is called the fundamental matrix or the transition matrix. It satisfies the matrix differential equation

$$\underline{\dot{\Phi}}(t,t_0) = \underline{F}(t)\ \underline{\Phi}(t,t_0) \tag{2.8a}$$

with initial conditions

$$\underline{\Phi}(t,t_0) = \underline{I} \ . \tag{2.8b}$$

The first term on the right-hand side of equation (2.7), the solution of equation (2.6), is called the complementary function or the transient solution. It gives the solution of the system of differential equations in absence of an input or forcing function $\underline{\underline{G}}\underline{w}(t)$. The second term is called the particular integral or the steady state solution. It adds to the solution the term originating from the functions $\underline{\underline{G}}\underline{w}(t)$ in equation (2.5a).

For a constant coefficient matrix F the transition matrix is given by the matrix exponential

$$\underline{\Phi}(t,t_0) = e^{\underline{F}(t-t_0)} \tag{2.9a}$$

or using the well-known series expansion of the exponential by

$$\underline{\Phi}(t,t_0) = \sum_{i=0}^{\infty} \frac{(t-t_0)^i \, \underline{F}^i}{i\,!} \tag{2.9b}$$

Using the Cayley-Hamilton theorem the infinite series expansion can be replaced by a finite series of order N where N is the rank of $\underline{F}$. A discussion of this approach and its usefulness in applications is given in Biermann (1971). However, it is not always the most direct way to a solution. Often the use of the Laplace transform is simpler.

The Laplace transform L of a function x(t) is defined as

$$f(s) = L\{x(t)\} \tag{2.10a}$$

$$= \int_0^{\infty} e^{-st} x(t)\,dt \tag{2.10b}$$

where $t > 0$ and the integral converges for some value of s. Similarly, for a vector $\underline{x}(t)$ of functions $x_i(t)$

$$\underline{f}(s) = L\{\underline{x}(t)\}$$

$$= \int_0^{\infty} e^{-st} \underline{x}(t)\,dt \tag{2.11b}$$

where the operation (2.10b) is now performed on each element of $\underline{x}(t)$. Taking the Laplace transform of equation (2.6) results in

$$s\underline{f}(s) - \underline{x}(0) = \underline{F}\;\underline{f}(s) \tag{2.12a}$$

and using (2.5b) we obtain

$$(s\underline{I} - \underline{F})\;\underline{f}(s) = \underline{c}\;. \tag{2.12b}$$

Thus, the system of differential equations (2.6) has been converted into a system of algebraic equations. If the inverse of $(s\underline{I} - \underline{F})$ exists we can write

$$\underline{f}(s) = (s\underline{I} - \underline{F})^{g}\underline{c}$$

where the superscript g denotes some suitably defined inverse. In the following, we will only use the Cayley inverse, thus obtaining

$$\underline{f}(s) = (s\underline{I} - \underline{F})^{-1}\underline{c} \ . \tag{2.12c}$$

Hochstadt (1975) discusses in detail the existence of the Cayley inverse for the above case.

The inverse Laplace transform L^{-1} reverses the operation (2.11), i.e.

$$\underline{x}(t) = L^{-1}\{\underline{f}(s)\} \tag{2.13}$$

where the same symbols as in equation (2.11) have been used. The functions x(t) and f(s) are often called a Laplace transform pair. Thus, the solutions to the algebraic equations (2.12c) can be inverse transformed to provide solutions to the original differential equations (2.6). This elegant and powerful technique will be frequently used in the following and therefore a simple example will be given here to demonstrate the salient features of the technique.

Consider the system of differential equations

$$\delta\dot{v} = -K\delta v - \omega^2\delta p$$
$$\delta\dot{p} = \delta v$$

which is of type (2.6) with

$$\underline{x} = \begin{bmatrix} \delta v \\ \delta p \end{bmatrix} \ , \quad \underline{F} = \begin{bmatrix} -K & -\omega^2 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \underline{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{2.14}$$

We thus have

$$(s\underline{I} - \underline{F}) = \begin{matrix} s+K & \omega^2 \\ -1 & s \end{matrix}$$

and its inverse

$$(s\underline{I} - \underline{F})^{-1} = \frac{1}{s(s+K) + \omega^2}\begin{bmatrix} s & -\omega^2 \\ 1 & s+K \end{bmatrix} \ .$$

The individual components of $\underline{f}(s)$ are therefore

$$f_1(s) = \frac{c_1 s}{s(s+K) + \omega^2} - \frac{c_2\omega^2}{s(s+K) + \omega^2}$$

$$= c_1 f_{11}(s) - c_2 f_{12}(s)$$

$$f_2(s) = \frac{c_1}{s(s+K) + \omega^2} + \frac{(s+K)c_2}{s(s+K) + \omega^2}$$

$$= c_1 f_{21}(s) + c_2 f_{22}(s)$$

Due to the linearity property of the inverse Laplace transform we can take the transforms of the individual terms. In a table of transform pairs, as e.g. McCullum and Brown (1965) or Doetsch (1971), we can find the inverse transform of $f_{21}(s)$

$$L^{-1}\{f_{21}(s)\} = L^{-1}\left\{\frac{1}{s^2(s+K) + \omega^2}\right\}$$

$$= \frac{1}{D} e^{-\frac{K}{2}t} \sin Dt = x_{21}(t) \tag{2.15c}$$

where $D = \sqrt{\omega^2 - \frac{K^2}{4}}$ and $D > 0$. Using the same transform pair and the linearity property we get

$$L^{-1}\{f_{12}(s)\} = L^{-1}\left\{\frac{\omega^2}{s(s+K) + \omega^2}\right\}$$

$$= \frac{\omega^2}{D} e^{-\frac{K}{2}t} \sin Dt = x_{12}(t) \tag{2.15b}$$

Then we have

$$L^{-1}\{f_{11}(s)\} = L^{-1}\{sf_{21}(s)\}$$

and using the formula

$$L^{-1}\{sf(s)\} = x'(t) + f(0)\delta(t)$$

we obtain

$$L^{-1}\{f_{11}(s)\} = x'_{21}(t) + f_{21}(0)\delta(t)$$

or

$$L^{-1}\left\{\frac{s}{s(s+K) + \omega^2}\right\} = \frac{1}{D} e^{-\frac{K}{2}t}\{ D \cos Dt - \frac{K}{2} \sin Dt\} + \frac{1}{\omega^2} \tag{2.15a}$$

$$= x_{11}(t) \; .$$

Finally $f_{22}(s)$ can bè split into the two components

$$f_{22}(s) = \frac{s}{s(s+K) + \omega^2} + \frac{K}{s(s+K) + \omega^2}$$

which can be Laplace transformed using equations (2.15a) and (2.15c)

$$L^{-1}\left\{\frac{(s+K)}{s(s+K) + \omega^2}\right\} = x_{11}(t) + Kx_{21}(t)$$

$$= \frac{1}{D} e^{-\frac{K}{2}t}\{\frac{K}{2}\sin Dt + D \cos Dt\} + \frac{1}{\omega^2} \tag{2.15d}$$

$$= x_{22}(t)$$

Thus the solution of equation (2.14) can be written as

$$\underline{x}(t) = \begin{bmatrix} \delta v \\ \delta p \end{bmatrix} = \begin{bmatrix} c_1 x_{11}(t) - c_2 x_{12}(t) \\ c_1 x_{21}(t) + c_2 x_{22}(t) \end{bmatrix} \tag{2.16}$$

where the terms $x_{11}(t)$ to $x_{22}(t)$ are given by formulas (2.15a) to (2.15d).

This simple example shows already that it is important to find the inverse Laplace transform of the determinant of the inverse matrix. The inverse Laplace transform of most elements of $(s\underline{I} - \underline{F})^{-1}$ can then be obtained by rather elementary manipulations. The example does not show that the analytical inversion of $(s\underline{I} - \underline{F})$ is a major problem. If the size of the matrix exceeds 10, it is often impossible to come up with manageable expressions.

The basic model (2.5) is not always sufficient to cover all cases encountered in practical problems. Two important extensions are the modelling of coloured noise and nonlinearities in the basic model. Time correlated noise processes will be used to model the effect of the anomalous gravity field. The usual approach is by state vector augmentation. The coloured noise $\underline{w}_1$ is modelled as the output of a linear system

$$\dot{\underline{w}}_1 = F_w \underline{w}_1 + \underline{w}_3 \tag{2.17}$$

where $\underline{w}_3$ is again white noise. This equation is added to equation (2.5) resulting in

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\underline{w}} \end{bmatrix} = \begin{bmatrix} F & G \\ 0 & F_w \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{w}_1 \end{bmatrix} + \begin{bmatrix} G & 0 \\ 0 & \underline{I} \end{bmatrix} \begin{bmatrix} \underline{w}_2 \\ \underline{w}_3 \end{bmatrix} \tag{2.18}$$

where $\underline{w}$ in equation (2.5) is defined by

$$\underline{w} = \underline{w}_1 + \underline{w}_2$$

and $\underline{w}_2$ is white noise. Because of equation (2.18) this is called state vector augmentation. The assumption is that the correlated noise process can indeed be modelled by (2.17). In theory, this is not always possible. In practice, it can usually be done with sufficient accuracy. In engineering applications the following F_w-matrices are often used:

$$F_w = -\beta \; ,$$

modelling a first-order Markov process, and

$$F_w = 0 \; ,$$

modelling a bias term. A more detailed discussion of suitable models for the anomalous gravity field will be given in section 4.

Nonlinearities in the basic model will play no role in the following and reference is therefore made to standard textbooks as Gelb (1974) and Eykhoff (1974).

3. ERROR PROPAGATION AND STEADY STATE ERRORS

There are two major practical advantages of using the dynamic model (2.5). First, the real-time implementation of these formulas including update measurements presents no problem. Kalman filtering and its alternatives have been investigated in great detail and their numerical characteristics are well-known. Second, a differential equation for the error propagation can be derived from which an expression for the steady state error of individual states can be obtained. The derivation of the basic error equation, the linear variance equation, and its solution for the steady state errors will be given in this section. The application of this technique to derive analytical expressions for the position errors in airborne inertial navigation will be given in section 5.

The statistical description of the total error of a dynamical system at a certain time t is given by the covariance matrix P(t)

$$\underline{P}(t) = E\{\underline{x}(t)\ \underline{x}^T(t)\}\ . \tag{3.1}$$

The change of $\underline{P}(t)$ with time models the error propagation. We will thus consider the function $\underline{P}(t, t-\varepsilon)$

$$\underline{P}(t, t-\varepsilon) = E\{\underline{x}(t)\ \underline{x}^T(t-\varepsilon)\} \tag{3.2}$$

where ε is a small positive number, and then use limit arguments to arrive at the result. Differentiation of equation (3.2) with respect to time yields

$$\dot{\underline{P}}(t, t-\varepsilon) = E\{\dot{\underline{x}}(t)\underline{x}^T(t-\varepsilon) + \underline{x}(t)\dot{\underline{x}}^T(t-\varepsilon)\}\ .$$

Using equation (2.5a) we obtain

$$\dot{\underline{P}}(t, t-\varepsilon) = E\{[\underline{\underline{F}}\underline{x}(t) + \underline{\underline{G}}\underline{w}(t)]\underline{x}^T(t-\varepsilon) + \underline{x}(t)[\underline{x}^T(t-\varepsilon)\underline{\underline{F}}^T + \underline{w}^T(t-\varepsilon)\underline{\underline{G}}^T]\}$$

or

$$\dot{\underline{P}}(t, t-\varepsilon) = E\{\underline{\underline{F}}\underline{x}(t)\underline{x}^T(t-\varepsilon) + \underline{\underline{G}}\underline{w}(t)\underline{x}^T(t-\varepsilon) + \underline{x}(t)\underline{x}^T(t-\varepsilon)\underline{\underline{F}}^T + \underline{x}(t)\underline{w}^T(t-\varepsilon)\underline{\underline{G}}^T\}\ . \tag{3.3}$$

Because of equation (2.4a) the statistical expectation of the second term in equation (3.3) must be zero. The first and third term are easy to evaluate. The last term is rewritten using equation (2.7) for $\underline{x}(t)$ and equation (2.5b) for $\underline{c}$

$$E\{\underline{x}(t)\underline{w}^T(t-\varepsilon)\underline{G}^T\} = E\{\underline{\Phi}(t,o)\underline{x}(0)\underline{w}^T(t-\varepsilon) + \int_0^t \underline{\Phi}(t,s)\underline{G}\underline{w}(s)\underline{w}^T(t-\varepsilon)\underline{G}^T ds\} .$$

Again, the evaluation of the first term results in zero because of equation (2.4a) while the second term can be written as

$$E\{\underline{x}(t)\underline{w}^T(t-\varepsilon)\underline{G}^T\} = \underline{\Phi}(t, t-\varepsilon)\, \underline{G}\,\underline{Q}\,(t-\varepsilon)\underline{G}^T$$

using equation (2.2a). Thus we obtain

$$\dot{\underline{P}}(t, t-\varepsilon) = E\{\underline{F}\underline{x}(t)\underline{x}^T(t-\varepsilon) + \underline{x}(t)\underline{x}^T(t-\varepsilon)\underline{F}^T + \underline{\Phi}(t, t-\varepsilon)\, \underline{G}\,\underline{Q}\,(t-\varepsilon)\underline{G}^T\} \tag{3.4}$$

Using $\varepsilon \to 0$, equation (3.1) and

$$\underline{\Phi}(t,t) = I$$

we get

$$\dot{\underline{P}}(t) = \underline{F}\,\underline{P}(t) + \underline{P}(t)\,\underline{F}^T + \underline{G}\,\underline{Q}(t)\,\underline{G}^T \tag{3.5}$$

which is the required expression. Its name, linear variance equation, is due to the fact that it is linear in $\underline{P}$. If update measurements (2.5c) are made, a similar derivation leads to the matrix equation

$$\dot{\underline{P}}(t) = \underline{F}\,\underline{P}(t) + \underline{P}(t)\,\underline{F}^T + \underline{G}\,\underline{Q}(t)\,\underline{G}^T - \underline{P}(t)\,\underline{H}\,\underline{R}^{-1}\underline{H}^T\underline{P}^T(t) \tag{3.6}$$

which obviously is nonlinear in $\underline{P}$. This equation is known as the matrix Riccati equation and has been thoroughly investigated in the mathematical literature in connection with variational problems. Its solution is generally obtained by transforming the nonlinear equation into a set of two linear equations of the form

$$\begin{bmatrix} \dot{\underline{X}}(t) \\ \dot{\underline{Y}}(t) \end{bmatrix} = \begin{bmatrix} -\underline{F}^T & \underline{H}^T\underline{R}^{-1}\underline{H} \\ \underline{G}\underline{Q}\underline{G}^T & \underline{F} \end{bmatrix} \begin{bmatrix} \underline{X}(t) \\ \underline{Y}(t) \end{bmatrix} \tag{3.7a}$$

with initial values

$$\underline{X}(t_0) = \underline{I}$$
$$\underline{Y}(t_0) = \underline{P}(t_0) \ . \qquad (3.7b)$$

The matrices $\underline{Y}$ and $\underline{X}$ are related via

$$\underline{Y}(t) = \underline{P}(t)\ \underline{X}(t) \qquad (3.8)$$

from which $\underline{P}(t)$ can be obtained

$$\underline{P}(t) = \underline{Y}(t)\ \underline{X}^{-1}(t) \ . \qquad (3.9)$$

These transformations can be verified by differentiating equation (3.9) and transforming it into the second equation (3.7a) by use of equation (3.6) and the first row in (3.7a). For details, see Liebelt (1967). $\underline{X}(t)$ is the transition matrix of the differential equation

$$\dot{\underline{X}}(t) = \{-\underline{F}^T + \underline{H}^T\underline{R}^{-1}\underline{H}\,\underline{P}\,(t)\}\ \underline{X}(t) \qquad (3.10)$$

which is the *adjoint of the differential equation* (3.6). Since the inverse of a transition matrix always exists, the validity of equation (3.9) is secured.

To obtain the matrices $\underline{X}$ and $\underline{Y}$, we write the formal solution of the system (3.7a) by way of a transition matrix $\Phi(t,t_0)$ as

$$\begin{bmatrix} \underline{X}(t) \\ \underline{Y}(t) \end{bmatrix} = \underline{\Phi}(t,t_0) \begin{bmatrix} \underline{X}(t_0) \\ \underline{Y}(t_0) \end{bmatrix}$$

and using equation (3.7b) and partitioning

$$\begin{bmatrix} \underline{X}(t) \\ \underline{Y}(t) \end{bmatrix} = \begin{bmatrix} \underline{\Phi}_{11}(t,t_0) & \underline{\Phi}_{12}(t,t_0) \\ \underline{\Phi}_{21}(t,t_0) & \underline{\Phi}_{22}(t,t_0) \end{bmatrix} \begin{bmatrix} \underline{I} \\ \underline{P}(t_0) \end{bmatrix} \qquad (3.11)$$

where again

$$\dot{\underline{\Phi}}(t,t_0) = \begin{bmatrix} -\underline{F}^T & \underline{H}^T\underline{R}^{-1}\underline{H} \\ \underline{GQG}^T & \underline{F} \end{bmatrix} \underline{\Phi}(t,t_0) \qquad (3.12)$$

The dimensions of the submatrices $\underline{\Phi}_{ij}$ $(i,j = 1,2)$ can immediately be obtained from equation (3.12). Using equation (3.9) and (3.11) we finally get the solution

$$\underline{P}(t) = \{\underline{\Phi}_{21}(t,t_0) + \underline{\Phi}_{22}(t,t_0)\underline{P}(t_0)\}\{\underline{\Phi}_{11}(t,t_0) + \underline{\Phi}_{12}(t,t_0)\underline{P}(t_0)\} \qquad (3.13)$$

which gives the covariance matrix $\underline{P}$ at an arbitrary time t in terms of its initial values $\underline{P}(t_0)$ and the transition matrices of the system (3.7a). This solution goes back to Reid (1946). A proof and general discussion is given in Levin (1959). Its first application to optimal estimation problems can be found in Kalman and Bucy (1961).

The practical use of equation (3.13) is somewhat limited because the transition matrices are seldom given in analytical form. Usually, they are computed by numerical integration and in this case recurrence relations are much more efficient to use. In our case, however, equation (3.13) can be sufficiently simplified to obtain useful analytical expressions. The first simplification is due to the fact that we do not have to solve the general equation (3.6) but the special case (3.5). Noting that

$$\underline{H}^T\underline{R}^{-1}\underline{H} = \underline{0}$$

and using the Laplace transform technique on equation (3.12), we obtain

$$\underline{\Phi}(t,t_0) = L^{-1}\left\{\begin{bmatrix} s\underline{I} + \underline{F}^T & \underline{0} \\ -\underline{G}\,\underline{Q}\,\underline{G}^T & s\underline{I} - \underline{F} \end{bmatrix}^{-1}\right\}$$

and after performing the matrix inversion

$$\underline{\Phi}(t,t_0) = L^{-1}\left\{\left[\begin{array}{c|c} (s\underline{I} + \underline{F}^T)^{-1} & \underline{0} \\ \hline (s\underline{I} - \underline{F})^{-1}\,\underline{G}\,\underline{Q}\,\underline{G}^T\,(s\underline{I} + \underline{F}^T)^{-1} & (s\underline{I} - \underline{F})^{-1} \end{array}\right]\right\} \qquad (3.14)$$

and for the individual submatrices

$$\underline{\Phi}_{11}(t,t_0) = L^{-1}\{(s\underline{I} + \underline{F}^T)^{-1}\} \qquad (3.15a)$$

$$\underline{\Phi}_{12}(t,t_0) = 0 \qquad (3.15b)$$

$$\underline{\Phi}_{21}(t,t_0) = L^{-1}\{(s\underline{I} - \underline{F})^{-1}\underline{G}\,\underline{Q}\,\underline{G}^T(s\underline{I} + \underline{F}^T)^{-1}\} \tag{3.15c}$$

$$\underline{\Phi}_{22}(t,t_0) = L^{-1}\{(s\underline{I} - \underline{F})^{-1}\} \ . \tag{3.15d}$$

Furthermore, we are only interested in the steady state errors, i.e. in that part of $\underline{P}(t)$ which is independent of $\underline{P}(t_0)$. Thus, observing equation (3.15b) and the steady state requirement, equation (3.13) simplifies to

$$\underline{P}_{ss}(t) = \Phi_{21}(t,t_0)\ \Phi_{11}^{-1}(t,t_0) \ . \tag{3.16}$$

where the subscripts 'ss' denote the steady state solution. The matrix identity

$$(sI + F^T)^{-1} = \{sI + F)^{-1}\}^T \tag{3.17}$$

can be applied to show that

$$\Phi_{22}^T(t,t_0) = \Phi_{11}^{-1}(t,t_0) \tag{3.18}$$

and is also useful in simplifying the algebraic operations in (3.15c). The diagonal terms of $\underline{P}_{ss}$ will be used in section 5 to derive analytical expressions for the steady state position error.

4. COVARIANCE MODELS FOR THE ANOMALOUS GRAVITY FIELD

In order to apply the error analysis developed in the last section, the effects of the anomalous gravity field have to be modelled into a form compatible with equation (2.5). This requires, first of all, that the characteristical features of the field can be described in statistical language, e.g. in terms of random functions and their associated covariance functions. The justification of such an approach is given in Moritz (1980) and within the limits indicated there, it is applicable to the following analysis. The second requirement is the transformation of the position dependent covariance function into a time dependent covariance function by way of the velocity. As long as the velocity is constant and the basic covariance function isotropic, this transformation corresponds to a simple "time-scaling" of distances between points. Since a constant aircraft velocity is a reasonable assumption in our case, and since all gravity anomaly covariance functions in practical use are isotropic, this transformation does not present a problem. More important for the model selection are two other requirements. One is that the model is self-consistent, the other that it can be expressed by a set of differential equations of the form (2.17). Self-consistency means that the mathematical relationships which are known to exist between the anomalous potential and linear functionals of it have to be used when deriving the covariance functions of these quantities. In other words, the covariance functions of the potential and the covariance functions of its derivatives are not independent. In the geodetic literature self-consistency with respect to the covariance functions is usually discussed in the context of covariance propagation. The other requirement is obviously important whenever an error model of type (2.5) is used. It has found considerable attention in the navigation literature but has so far been neglected by geodesists. For our analysis both aspects are equally important.

When comparing the resulting covariance models, two major groups can be distinguished. They will be labeled 'geodetic' and

'navigational' to indicate their predominant user groups. The geodetic models are based on spherical approximation and are self-consistent and global in character. Upward continuation of these functions is no problem and the derivation of local models from the global function is relatively simple. The important parameters of these models are derived from different data groups, specifically spherical harmonic coefficients obtained from satellite observations, gravity anomalies and second-order gradients. Their major disadvantage for an error analysis of this type is that there is no simple way to express them in the form (2.17). A detailed discussion of the mathematical structure of these functions is contained in Tscherning (1976) and Moritz (1976). The fitting of a global function is described in Tscherning and Rapp (1974) and Jekeli (1978) while local aspects are treated in Schwarz and Lachapelle (1980) and Lachapelle and Schwarz (1980). Efficient numerical methods to compute the covariance function are treated in Sünkel (1979, 1980). Some modelling aspects of nonhomogeneous global covariance functions are discussed in Rummel and Schwarz (1977). The navigational models are based on a flat earth approximation and are local in character. Their major advantage is that they can be cast into the form (2.17), i.e. they can be represented as the output of a linear system driven by white noise. Not all of them are self-consistent and most of them cannot be upward continued in a simple way. Usually the upward continuation integral has to be evaluated. The extension of the local function to a global function is in general not possible. The important parameters of these functions are usually determined from one data group only, specifically gravity anomalies or deflections of the vertical.
Levine and Gelb (1969) were the first to model deflections of the vertical in this way. Subsequent contributions are due to Shaw et al (1969), Vyskosil (1970), Grafarend (1971), Kasper (1971) but it was especially Jordan (1972, 1973, 1981 et al) who in a series of papers discussed both the geodetic and navigational aspects of different models and synthesized the work of his predecessors.

This brief discussion shows already that for the airborne case a combination of the two approaches is needed. On the one hand, the geodetic model provides a simple procedure to compare different approximations of the gravity field in a consistent manner and to evaluate the effect of upward continuation. On the other hand, only the navigational model can be used in a state space formulation which forms the basis for the dynamic error model. All covariance functions will therefore in the sequel be based on a consistent geodetic model and only in the last step the transition to the navigational model will be made. This last step cannot be done rigorously but will always involve an approximation. To keep approximation errors small, the fitting of the navigational model will be done by way of the essential parameters defined by Moritz (1976). Agreement of these parameters for two models secures agreement of the error estimates derived from them. The remainder of this section will review the formulation of the geodetic model, define the essential parameters, and discuss some simple navigational models which can be used as approximations.

The rotation invariant, spatial covariance function of the anomalous potential is given by the general expression

$$K(P,Q) = A \sum_{n=1}^{\infty} k_n \left[\frac{R^2}{r_P r_Q}\right]^{n+1} P_n(\cos \psi_{PQ}) \tag{4.1}$$

where

- P,Q ... are two points in space outside the sphere with radius R, and with origin in the centre of gravity of the earth
- k_n ... are positive coefficients, i.e. $k_n \geq 0$
- r_P, r_Q ... are radial distances from the centre of the sphere
- P_n ... are the Legendre polynomials
- ψ_{PQ} ... is the spherical distance between P and Q.

K(P,Q) is a function of positive type, symmetric and harmonic in the space outside the sphere R and regular at infinity; for a discussion, see Krarup (1969). The coefficients k_n can be defined rather arbitrarily as long as their positivity is secured. However, the number of choices is considerably reduced when asking for coefficients which result in a simple closed expression for equation (4.1). This requirement is necessary for any practical application and has led to coefficients of the general type

$$k_n = \prod_{j=0}^{i} (n + b_j)^{-1}$$

where Π denotes the multiplication of the i factors on the right-hand side. Well-known examples are

$i = 1 \quad b_0 = -2 \quad b_1 = -1$

or $i = 2 \quad b_0 = -2 \quad b_1 = -1 \quad b_2 = 24$

resulting in coefficients

$$k_n = \frac{1}{(n-1)(n-2)} \tag{4.3a}$$

and

$$k_n = \frac{1}{(n-1)(n-2)(n+24)} \quad . \tag{4.3b}$$

By expressing the k_n as a sum of partial fractions the infinite sum in equation (4.1) can be expressed as a sum of closed analytical expressions F_i resulting from the summation of the corresponding i infinite series. As an example the covariance function of the model (4.3b) can be written as

$$K(P,Q) = \frac{AR^2}{650} \{25\ F_{-2} - 26[F_{-1} - s^3 P_2(t)] + F_{24} - \frac{s}{24} - \frac{s^2 t}{25} - \frac{s^3 P_2(t)}{26}\} \tag{4.4}$$

where

$$F_{-1} = s(M + st\ \ell n \frac{2}{N}) \tag{4.5a}$$

$$F_{-2} = \frac{1}{2} s\{M(1 + 3st)\} + s^3 P_2(t)\ \ell n \frac{2}{N} + \frac{1}{4} s^3 (1 - t^2) \tag{4.5b}$$

and F_{24} can be computed from the recursion formula

$$F_{k+1} = \frac{1}{k \cdot s} \{L + (2k-1)tF_k - (k-1)s^{-1}F_{k-1}\} . \qquad (4.5c)$$

The auxiliary quantities L,M,N are defined by

$$L = \{1 - 2st + s^2\}^{\frac{1}{2}}$$
$$M = 1 - L - st$$
$$N = 1 + L - st$$

where

$$t = \cos \psi_{PQ}$$
$$s = \frac{R^2}{r_P r_Q}$$

and A and $P_2(t)$ have been defined earlier. For a detailed derivation reference is made to Tscherning and Rapp (1974). Expressions of the form (4.5a) to (4.5c) are easy to evaluate on a computer and are simple enough to allow the formation of all derivatives necessary for covariance propagation. Thus, the basic model given by equations (4.1) and (4.2) is used to derive a self-consistent set of covariance functions for functionals of the anomalous potential. These derivations are given in Tscherning and Rapp (1974) and Tscherning (1976) for a number of different models.

To obtain a satisfactory fit of empirically determined data to equation (4.1), the following variables can be changed:

- the size of A
- the radius of the sphere R
- the coefficients k_n according to one of the models (4.2).

The effect of a change in the first variable is simply that of a scaling factor for all coefficients and thus for the total covariance function. It is directly related to the variance K(0,0) or the derived variance of the gravity anomaly C(0,0) which can be easily determined from given point gravity anomalies. A change of R affects the spectrum of K(P,Q) in a non-uniform way. For instance, a decrease of R works similar to a low pass filter by strongly damping the higher frequencies. Similarly, when going from model (4.3a) to model (4.3b) a damping of the high degree terms will be the result. It is therefore important to ensure that a covariance function of type (4.1) fits well to data representing different parts of the spectrum. Typically, geoidal undulations or geopotential models from satellite observations contain mainly low frequency information while torsion balance measurements or gradiometer data represent the high frequency range. Gravity anomalies and deflections of the vertical take an intermediate position containing mostly information on the medium frequency range. At present it appears that a model of type (4.3b) or a combination of this model with one of a simpler structure gives the best fit to all available data. It was therefore decided to use model (4.3b) for the following computations. Once the basic structure of the model is given, we have to specify what will be understood by goodness of fit. For instance, is it important to approximate an empirical covariance function as closely as possible everywhere or are certain regions especially important? Or, should empirical degree variances computed from satellite solutions replace the low order $\{Ak_n\}$ in equation (4.1)? Questions of this type have been answered by Moritz (1976, 1977) in a discussion of the essential parameters of planar covariance functions. These functions can be considered as planar equivalents of the covariance functions (4.1) and results obtained for them carry over to the spherical case with only small modifications.

Moritz (1976) defines three essential parameters for covariance functions of this type: the variance K_0, the correlation length ζ, and the curvature parameter χ. The variance K_0 is the value of the covariance function $K(\psi)$ for the argument $\psi = 0$, i.e.

$$K_0 = K(0) \qquad (4.6a)$$

The correlation length is the value of the argument for which

$$K(\zeta) = \frac{1}{2} K_0 \ . \qquad (4.6b)$$

The curvature parameter χ is related to the curvature κ of $K(\psi)$ at $\psi = 0$ by

$$\chi = \kappa \cdot \zeta^2/K_0 \tag{4.6c}$$

where κ is given in the usual way by

$$\kappa = \frac{K''}{(1 + K'^2)^{3/2}} \tag{4.7}$$

and

$$K' = \frac{\partial K}{\partial \rho} \qquad K'' = \frac{\partial^2 K}{\partial \rho^2}$$

with

$$\rho = R\psi \, .$$

Usually, the fit is made to empirical values of the gravity anomaly covariance function $C(\psi)$ because large data samples are available. In this case, the curvature κ is equal to G_0 , the variance of the horizontal gradients of Δg . Thus we have

$$\chi = G_0\zeta^2 / C_0 \quad ; \tag{4.8}$$

for a derivation see Moritz (1976). In this case all three essential parameters can be determined from actual measurements. C_0 is the global variance of the gravity anomalies; ζ is the correlation length of the empirical covariance function $C(\psi)$ which can be determined graphically; G_0 can be computed either from dense point gravity anomalies or from torsion balance or gradiometer measurements. For a more detailed discussion, see Schwarz and Lachapelle (1980).

Two covariance functions which have the same essential parameters are not necessarily equal everywhere. However, their general behaviour with respect to interpolation will be similar. This is obviously a very important property for the analysis in this report. It can also be shown that the shape of the covariance curve at distances larger than about $1.5\,\zeta$ does not influence the interpolation very much. This is quite important for any empirical covariance fitting.

Besides relating the covariance model (4.1) to the actual data, the set of essential parameters can also be used to get a good fit between the geodetic and the navigational models. This will be the main application in the following. Covariance functions for the deflections of the vertical will be derived from the best available model (4.1). Upward continuation in this model requires only a change of r_P and r_Q. Improved gravity models can be simulated by subtracting a certain number of low degree coefficients from the series. By basing all computations on one fundamental model of type (4.1), self-consistency between different functionals of T as well as consistency between the different cases is secured. By defining each covariance function in terms of its essential parameters, self-consistency of the approximating navigational model is not required anymore. Emphasis will therefore be on selecting navigational models which give a best fit in terms of essential parameters for the individual covariance functions.

The navigational covariance models most frequently used are low-order Gauss-Markov models whose covariance functions are of the general form (Gelb, 1974)

$$K(\tau) = \sigma_n^2 \, e^{-\beta_n|\tau|} \sum_{k=0}^{n-1} \frac{\Gamma(n)(2\beta_n|\tau|)^{n-k-1}}{(2n-2)!\;k!\;\Gamma(n-k)} + m_n^2 \tag{4.9}$$

where

σ_n^2 ... is the variance of the process

β_n ... is a parameter related to the correlation length

m ... is the mean

$\Gamma(n)$ ... is the Gamma function

n ... is the order of the Gauss-Markov process

and $|\tau|$ is the independent variable, i.e. distance in geodetic applications and time in navigation applications. Implicit in this definition is that the process is stationary and its probability density function is Gaussian. Processes of this type are defined by a stochastic differential equation of order n or equivalently by a set of n first-order stochastic differential equations. They are therefore convenient to use in our problem.

The covariance function of the first-order Gauss-Markov process is the simple exponential function

$$K(\tau) = \sigma_1^2 \, e^{-\beta_1|\tau|} + m_1^2 \quad . \tag{4.10}$$

The process is defined by the differential equation

$$\dot{\underline{x}} + \beta_1 x = \underline{w} \tag{4.11}$$

where w is white noise. Because of its simplicity, this process is often used to model physical phenomena. It is not a consistent model with respect to the gravity field. But since consistency is provided by the geodetic model this is not a major concern. The essential parameters of this process are

$$\left.\begin{aligned} K_0 &= \sigma_1^2 \\ \zeta_1 &= -\frac{1}{\beta_1} \ln \frac{1}{2} = \frac{1}{\beta_1} \ln 2 = \frac{0.693}{\beta_1} \\ \chi_1 &= (\ln 2)^2 = 0.480 \end{aligned}\right\} \tag{4.12a}$$

It should be noted that in the navigational literature the correlation distance $\bar{\zeta}$ is used instead of ζ. It is defined as

$$K(\bar{\zeta}) = \frac{1}{e} K_0 \tag{4.13}$$

where e is the basis of the natural logarithm. In order to differentiate between ζ and $\bar{\zeta}$ without too much verbal acrobatics we will continue to call ζ the correlation length while $\bar{\zeta}$ will be called the correlation distance. For the above case

$$\bar{\zeta}_1 = \frac{1}{\beta_1} \quad . \tag{4.12b}$$

The covariance function of the second-order Gauss-Markov process is given by

$$K(\tau) = \sigma_2^2 \{1 + \beta_2|\tau|\} \, e^{-\beta_2|\tau|} + m^2 \; . \tag{4.14}$$

The corresponding differential equation is of the form

$$\ddot{\underline{x}} + 2\beta_2\dot{\underline{x}} + \beta_2^2\underline{x} = \underline{w} \tag{4.15}$$

The essential parameters are

$$\left.\begin{aligned} K_0 &= \sigma_2^2 \\ \zeta_2 &= \frac{1.678}{\beta_2} \\ \bar{\zeta}_2 &= \frac{2.146}{\beta_2} \\ \chi_2 &= 2.817 \end{aligned}\right\} \tag{4.16}$$

where $\bar{\zeta}_2$ is obtained by solving the equation

$$\ln(1 + \beta_2\bar{\zeta}_2) = \beta_2\bar{\zeta}_2 - 1$$

and ζ_2 by solving

$$\ln(1 + \beta_2\zeta_2) = \beta_2\zeta_2 - \ln 2 \ .$$

Similarly, the covariance function of the third-order Gauss-Markov process is given by

$$K(\tau) = \sigma_3^2 \{1 + \beta_3|\tau| + \frac{1}{3}\beta_3^2 |\tau^2|\} e^{-\beta_3(\tau)} + m^2 \ . \tag{4.17}$$

with the corresponding differential equation

$$\dddot{\underline{x}} + 3\beta_3\ddot{\underline{x}} + 3\beta_3^2\dot{\underline{x}} + \beta_3^3\underline{x} = \underline{w} \tag{4.18}$$

and the essential parameters

$$\left.\begin{aligned} K_0 &= \sigma_3^2 \\ \zeta_3 &= \frac{2.330}{\beta_3} \\ \bar{\zeta}_3 &= \frac{2.903}{\beta_3} \\ \chi_3 &= 1.810 \end{aligned}\right\} \tag{4.19}$$

The correlation distance $\bar{\zeta}$ is computed by solving

$$\ln\,(1 + \bar{\zeta}_3\beta_3 + \frac{1}{3}\,\bar{\zeta}_3^{\,2}\,\beta_3^{\,2}\,) = \bar{\zeta}_3\beta_3 - 1$$

and the correlation length ζ by solving

$$\ln\,(1 + \zeta_3\beta_3 + \frac{1}{3}\,\zeta_3^{\,2}\,\beta_3^{\,2}\,) = \zeta_3\beta_3 - \ln 2\ .$$

Both the second and the third-order Gauss-Markov models are self-consistent models for the representation of the anomalous gravity field. Jordan (1972) advocates the third-order model for the anomalous potential because it results in a gravity anomaly covariance function with a realistic behaviour at $\psi = 0$ or $\tau = 0$ respectively. As mentioned above, self-consistency of the navigational model is not a major concern in our case. The selection of an appropriate model should therefore be based on the agreement between the essential parameters of the geodetic model and the navigational model. Equations (4.12a), (4.16) and (4.19) show that variance and correlation length are variable while the curvature parameter is constant. This means that any parameters K_0, ζ, required by the geodetic model, can be accommodated by the navigational model. The selection of an optimal fit can therefore be done on the basis of χ only. In our case the curvature parameters of the deflection covariance functions $C(\xi,\xi)$ and $C(\eta,\eta)$ are required. They can be derived from equation (4.1) by using the well known relations

$$\xi = -\frac{1}{r\gamma}\frac{\partial T}{\partial \phi}$$

$$\eta = -\frac{1}{r\gamma\cos\phi}\;\frac{\partial T}{\partial \lambda}$$

to obtain

$$\mathrm{cov}(\xi_P,\xi_Q) = \frac{1}{r_P\;r_Q}\;\frac{1}{\gamma_P\;\gamma_Q}\;\frac{\partial^2 K(P,Q)}{\partial\phi_P\;\partial\phi_Q} \qquad (4.20a)$$

$$\mathrm{cov}(\eta_P,\eta_Q) = -\frac{1}{r_P r_Q \cos\phi_Q}\,\frac{1}{\gamma_P\,\gamma_Q}\,\frac{\partial^2 K(P,Q)}{\partial\lambda_P\,\partial\lambda_Q} \tag{4.20b}$$

The differentiations of K(P,Q) can be done by using

$$\cos\psi_{PQ} = \sin\phi_P\,\sin\phi_Q + \cos\phi_P\,\cos\phi_Q\,\cos(\lambda_Q - \lambda_P)$$

and formulas (4.4) and (4.5). Then introducing again planar approximation, a double differentiation of the functions (4.20) with respect to ρ will give κ and thus χ. To avoid the cumbersome differentiations, it has been assumed that the curvature parameter of the gravity anomaly covariance functions is close enough to the corresponding parameter of the deflection covariance function. Thus, equation (4.8) has been used. The actual computation and comparison of these parameters is given in section 6.

5. THE STEADY STATE POSITION ERROR INDUCED BY GRAVITY FIELD APPROXIMATIONS

The procedure outlined in section 3 led to the formula

$$\underline{P}_{ss}(t) = \underline{\Phi}_{21}(t,t_0)\,\underline{\Phi}_{11}^{-1}(t,t_0) \tag{3.16}$$

for the steady state errors of a linear system. Using equation (3.18) this can be rewritten as

$$\underline{P}_{ss}(t) = \underline{\Phi}_{21}(t,t_0)\,\underline{\Phi}_{22}^{T}(t,t_0) \tag{5.1}$$

To obtain the steady state position errors induced by different gravity field approximations, the system of differential equations for this problem has to be set up. For a three-axes inertial navigation system the minimum system is of order 12 containing the following elements in the state vector

$$\underline{x}^T = (\varepsilon_N, \varepsilon_E, \varepsilon_D, \delta p_N, \delta p_E, \delta p_D, \delta v_N, \delta v_E, \delta v_D, \xi, \eta, \delta N) \tag{5.2}$$

where the subscripts N, E, D denote the directions of the three axes of a local-level system (North, East, Down) and

ε ... are attitude errors

δp ... are position errors

δv ... are velocity errors

ξ, η ... are the deflections of the vertical in north and east direction

δN ... is the geoidal undulation.

The dynamics matrix for the first nine states can be found in Britting (1971) while for the states representing the anomalous gravity field one of the models in section 4 can be used. The solution of this system applying Laplace transform techniques is a formidable task. To give an idea of the type of analytical manipulations necessary, reference is made to Wong and Schwarz (1979) where the transition matrix $\underline{\Phi}_{22}$ has been derived for the first 9 states in equation (5.2). Considering that the three additional states will complicate the derivation considerably and that the determination of $\underline{\Phi}_{21}$ is much more complicated than that of $\underline{\Phi}_{22}$, it was decided to consider only the single axis case, i.e. to completely decouple the three channels. Results obtained from numerical integration techniques indicate that this simplified case represents the error behaviour of the more general model quite well. Thus results obtained in this way will in general be also valid for the more extensive model. The obvious advantage is that a much smaller system has to be dealt with. This is probably the main reason why all previous investigations followed this route.

The single axis case for one of the horizontal channels is given by the differential equations

$$\delta\dot{v} = -K\delta v - \omega^2\delta p - \gamma\xi \tag{5.3a}$$

$$\delta\dot{p} = \delta v \tag{5.3b}$$

$$\dot{\xi} = -\beta_1\xi + u_1 \tag{5.3c}$$

where

γ ... is normal gravity which for this error analysis can always be replaced by

G ... the global mean value of gravity,

ω ... is the Schuler frequency

where

$$\omega = \left(\frac{G}{R}\right)^{\frac{1}{2}} \tag{5.4}$$

R ... is the mean radius of the earth

K ... is the feedback gain which is related to

d ... the loop damping parameter by

$$K = 2\,d\,\omega_s \,, \tag{5.5}$$

ξ ... is the deflection of the vertical (either ξ or η)

and u_1 ... is white noise defined by

$$E\{u_1(t)\;u_1(\tau)\} = q_1\delta(t-\tau) \tag{5.6}$$

where the power density spectrum in case of equation (5.3c) is given by

$$q_1 = 2\beta_1\;\sigma_1^2 \tag{5.7a}$$

and β_1 and σ_1 are defined as in equation (4.10). The power density spectrum for the second and third order Gauss-Markov process are given by

$$q_2 = 4\beta_2^3\ \sigma_2^2 \tag{5.7b}$$

and $q_3 = 16\beta_3^5\ \sigma_3^2$ (5.7c)

respectively. It should be noted that errorless velocity measurements have been assumed to avoid a mixing of error sources for the positioning error.

Putting equations (5.3) into the standard form (2.5) we have

$$\underline{x} = \begin{bmatrix} \delta v \\ \delta p \\ \xi \end{bmatrix} \qquad \underline{F} = \begin{bmatrix} -K & -\omega^2 & -G \\ 1 & 0 & 0 \\ 0 & 0 & -\beta_1 \end{bmatrix}$$

$$\underline{u} = \begin{bmatrix} 0 \\ 0 \\ u_1 \end{bmatrix} \qquad \underline{G} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Thus

$$(s\underline{I} - \underline{F}) = \begin{bmatrix} s+K & \omega^2 & G \\ -1 & s & 0 \\ 0 & 0 & s+\beta \end{bmatrix}$$

where the index of β has been omitted since no confusion is possible. Inversion of this matrix results in

$$(\underline{sI} - \underline{F})^{-1} = \begin{bmatrix} s/E_1 & -\omega^2/E_1 & -Gs/(s+\beta)E_1 \\ 1/E_1 & (s+K)/E_1 & -G/(s+\beta)E_1 \\ 0 & 0 & 1/(s+\beta) \end{bmatrix} \tag{5.8a}$$

where

$$E_1 = s^2 + sK + \omega^2 . \tag{5.8b}$$

Using equations (3.15d) and (3.18) we get

$$\underline{\Phi}_{22}(t,t_0) = L^{-1}\{(s\underline{I} - F)^{-1}\} = L^{-1}\{B\} \tag{5.9}$$

which gives us one of the required components in equation (5.1). To obtain $\underline{\Phi}_{21}$ we have first to form the matrix product

$$\bar{\underline{A}} = (s\underline{I} - \underline{F})^{-1}\ \underline{G}\,\underline{Q}\,\underline{G}^T\ (s\underline{I} + \underline{F}^T)^{-1}$$

which is of the following form

$$\bar{\underline{A}} = 2\sigma_1^2 \begin{bmatrix} -\dfrac{\beta G^2 s^2}{(s^2-\beta^2)E_1E_2} & \dfrac{\beta G^2 s}{(s^2-\beta^2)E_1E_2} & -\dfrac{\beta G s}{(s^2-\beta^2)E_1} \\ -\dfrac{\beta G^2 s}{(s^2-\beta^2)E_1E_2} & \dfrac{\beta G^2}{(s^2-\beta^2)E_1E_2} & -\dfrac{\beta G}{(s^2-\beta^2)E_1} \\ \dfrac{\beta G s}{(s^2-\beta^2)\ E_2} & -\dfrac{\beta G}{(s^2-\beta^2)\ E_2} & \dfrac{1}{(s^2-\beta^2)} \end{bmatrix} \tag{5.10a}$$

where

$$E_2 = s^2 - sK + \omega^2 \tag{5.10b}$$

and E_1 is again defined by equation (5.8b). $\underline{\Phi}_{21}$ is then obtained by the inverse Laplace transform

$$\underline{\Phi}_{21}(t,t_0) = L^{-1}\{\bar{\underline{A}}\} = 2\sigma_1^2\ L^{-1}\{\underline{A}\} \tag{5.11}$$

where $\underline{A}$ is the matrix on the right-hand side of equation (5.10a). The steady-state position error is determined by the second diagonal element in the $\underline{P}_{ss}$-matrix (5.1). Using the individual elements of the matrices $\underline{A}$ and $\underline{B}$ defined by equations (5.11) and (5.9) and the linearity property of the Laplace transform, we can write the variance σ_p^2 of the position error in terms of inverse Laplace transforms

$$\sigma_p^2 = 2\sigma_1^2\,[L^{-1}\{a_{21}\}L^{-1}\{b_{21}\} + L^{-1}\{a_{22}\}L^{-1}\{b_{22}\} + L^{-1}\{a_{23}\}L^{-1}\{b_{23}\}]\ . \tag{5.12}$$

The determination of the individual terms in equation (5.12) is most simply accomplished by using the complex inversion formula

$$x(t) = \frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega} e^{st}\,f(s)\,ds \qquad (5.13$$

where

$t > 0$

$s = u + iv$

$i = \sqrt{-1}$

and the integration is performed along a line $s = \gamma$ in the complex plane. The real number γ is chosen so that $s = \gamma$ lies to the right of all the singularities but is otherwise arbitrary. In practice, the integral is evaluated by considering the integral enclosed by the Bromwich contour C

$$\frac{1}{2\pi i} \oint_C e^{st} f(s)\, ds \tag{5.14}$$

which is depicted in Figure 5.1.

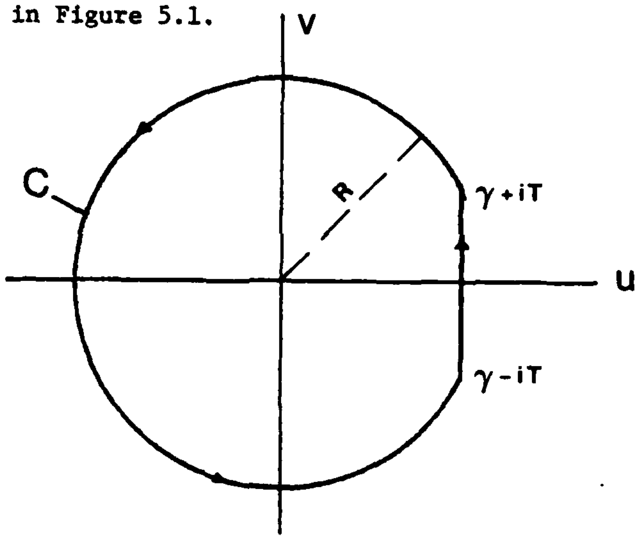

Fig. 5.1: The Bromwich contour

This integral is related to equation (5.13) by

$$x(t) = \lim_{R\to\infty} \left\{ \frac{1}{2\pi i} \oint_C e^{st} f(s)\, ds - \frac{1}{2\pi i} \oint_\Gamma e^{st} f(s)\, ds \right\} \tag{5.15}$$

where R is the radius of the circle and Γ the circular part of the contour C. A solution to (5.14) can be obtained using the residue theorem which states: If f(s) is analytic within and on the boundary of C of a region except at a finite number of poles p_j within the region having residues p_{-j} respectively, then

$$\oint_C f(s)\, ds = 2\pi i \sum_j p_{-j} \quad , \tag{5.16}$$

i.e. the solution can be obtained by determining the residues of f(s) at the poles p_j. They can be found by the general formula

$$p_{-j} = \lim_{s \to p_j} \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^n} \{(s-p_j)^n f(s)\} \tag{5.17}$$

where n is the order of the pole. For simple poles equation (5.17) reduces to

$$p_{-j} = \lim_{s \to p_j} \{(s-p_j)\ f(s)\} \quad . \tag{5.18}$$

Replacing f(s) by $e^{st} f(s)$ and assuming

$$\lim_{R \to \infty} \left\{ \frac{1}{2\pi i} \oint_\Gamma e^{st}\ f(s)\ ds \right\} = 0 \tag{5.19}$$

the solution of equation (5.15) can be written as

$$x(t) = \sum_j p_{-j} \quad . \tag{5.20}$$

where p_{-j} are the residues of $e^{st} f(s)$ at poles of f(s). Sufficient conditions under which equation (5.19) is correct are e.g. given in Doetsch (1971).

Returning to equation (5.12) we start with the first term

$$L^{-1}\{a_{21}\} = L^{-1} \left\{ - \frac{\beta G^2 s}{(s^2-\beta^2)E_1 E_2} \right\}$$

$$= L^{-1} \left\{ - \frac{\beta G^2 s}{(s^2-\beta^2)(s^2+sK+\omega^2)(s^2-sK+\omega^2)} \right\} .$$

E_1 and E_2 can be factored into

$$E_1 = (s+a)\ (s+b)$$
$$E_2 = (s-a)\ (s-b) \tag{5.21}$$

where

$$a = \frac{1}{2}K - \sqrt{\frac{1}{4}K^2 - \omega^2}$$

$$b = \frac{1}{2}K + \sqrt{\frac{1}{4}K^2 - \omega^2} \qquad (5.22a)$$

Because d < 1 in equation (5.5) we get the complex roots

$$a = \frac{1}{2}K - iD$$

$$b = \frac{1}{2}K + iD \qquad (5.22b)$$

where

$$D = \sqrt{\omega^2 - \frac{1}{4}K^2} \qquad (5.22c)$$

is now always positive. Using (5.21), we can write the inverse Laplace transform of a_{21} as

$$L^{-1}\{a_{21}\} = L^{-1}\left\{- \frac{\beta G^2 s}{(s^2-\beta^2)(s^2-a^2)(s^2-b^2)}\right\} .$$

Thus, in this case the function f(s) has the six simple poles $\pm\beta$, $\pm a$, $\pm b$. Using equations (5.20) and (5.18) we obtain

$$x_{21}(t) = L^{-1}\{a_{21}\}$$

$$= \lim_{s\to -\beta}\left\{- \frac{\beta G^2 s\ e^{st}}{(s-\beta)(s^2-a^2)(s^2-b^2)}\right\}$$

$$+ \lim_{s\to \beta}\left\{- \frac{\beta G^2 s\ e^{st}}{(s+\beta)(s^2-a^2)(s^2-b^2)}\right\}$$

$$+ \lim_{s\to -a}\left\{- \frac{\beta G^2 s\ e^{st}}{(s^2-\beta^2)(s-a)(s^2-b^2)}\right\}$$

$$+ \lim_{s\to a}\left\{- \frac{\beta G^2 s\ e^{st}}{(s^2-\beta^2)(s+a)(s^2-b^2)}\right\}$$

$$+ \lim_{s\to -b}\left\{- \frac{\beta G^2 s\ e^{st}}{(s^2-\beta^2)(s^2-a^2)(s-b)}\right\}$$

$$+ \lim_{s\to b}\left\{- \frac{\beta G^2 s\ e^{st}}{(s^2-\beta^2)(s^2-a^2)(s+b)}\right\}$$

Evaluating these expressions leads to

$$L^{-1}\{a_{21}\} = -\beta G^2 \left\{ \frac{\cos h\ \beta t}{(\beta^2-a^2)(\beta^2-b^2)} + \frac{\cos h\ at}{(a^2-\beta^2)(a^2-b^2)} \right.$$

$$\left. + \frac{\cos h\ bt}{(b^2-\beta^2)(b^2-a^2)} \right\} \qquad (5.23a)$$

A check of this result can be obtained by splitting a_{21} into

$$a_{21}(s) = f_1(s) \cdot f_2(s)$$

where

$$f_1(s) = \frac{s}{(s^2-a^2)(s^2-b^2)}$$

$$f_2(s) = \frac{\beta}{s^2-\beta^2} \ ,$$

finding the inverse Laplace transforms $x_1(t)$ and $x_2(t)$ from a table and applying the convolution theorem

$$x_{21}(t) = x_1(t) * x_2(t)$$

$$= \int_0^t x_1(\tau)\ x_2(t-\tau)\ d\tau \ .$$

This is, however, a more laborious procedure.

The next term, the inverse Laplace transform of a_{22} can be obtained by observing that

$$L^{-1}\{a_{22}\} = -L^{-1}\{ \frac{1}{s} a_{21}\} \ .$$

Since in general

$$L^{-1}\{ \frac{1}{s} f(s)\} = \int_0^t x(\tau)\ d\tau$$

where x(t) is the inverse transform of f(s), we have in our case

$$L^{-1}\{a_{22}\} = -\int_0^t x_{21}(\tau)\ d\tau$$

where $x_{21}(t)$ is given by the right-hand side of equation (5.23a). The integrals are simple to evaluate resulting in

$$L^{-1}\{a_{22}\} = \beta G^2 \left\{ \frac{\sin h\ \beta t}{\beta(\beta^2-a^2)(\beta^2-b^2)} + \frac{\sin h\ at}{a(a^2-\beta^2)(a^2-b^2)} + \frac{\sin h\ bt}{b(b^2-\beta^2)(b^2-a^2)} \right\} \tag{5.23b}$$

A check is again possible by use of the complex inversion formula and the residue theorem. Applying similar techniques for the other four terms, we obtain

$$L^{-1}\{a_{23}\} = \frac{\beta G}{(a^2-\beta^2)(b^2-\beta^2)} \left\{ (a+\beta)\cos h\ \beta t - \frac{(\beta^2+ab)\sin h\ \beta t}{\beta} + \frac{(a^2-\beta^2)\ e^{-bt}}{(a-b)} - \frac{(b^2-\beta^2)\ e^{-at}}{a-b} \right\} \tag{5.23c}$$

$$L^{-1}\{b_{12}\} = \frac{1}{D}\, e^{-\frac{1}{2}Kt} \sin Dt\ , \tag{5.23d}$$

compare equation (2.15c),

$$L^{-1}\{b_{22}\} = e^{-\frac{1}{2}Kt} \left\{ \frac{K \sin Dt}{2D} + \cos Dt \right\} \tag{5.23e}$$

and

$$L^{-1}\{b_{23}\} = -G \left\{ \frac{e^{-at}}{(b-a)(\beta-a)} + \frac{e^{-bt}}{(a-b)(\beta-b)} + \frac{e^{-\beta t}}{(a-\beta)(b-\beta)} \right\} \tag{5.23f}$$

The variance of the position error can now be obtained by performing the multiplications in formula (5.12) using the expressions (5.23a) to (5.23f). The somewhat lengthy formula is not written out here. It gives the steady state error as a function of time. Often we are only interested in the behaviour of the error after it has settled down, i.e. for $t \to \infty$. The relevant expressions have been published by several authors, see e.g. Jordan (1973), and are not rederived here. They are usually obtained by applying the final value theorem of Laplace transforms directly to equation (5.12).

For the first-order Gauss-Markov model with the covariance function (4.10) we obtain

$$\sigma^2_{P1} = \sigma_1^2 \; \frac{G^2}{K\omega^2} \; \frac{K + \beta_1}{(\beta_1^{\,2} + K\beta_1 + \omega^2)} \tag{5.24}$$

For the second-order Gauss-Markov model with the covariance function (4.14) we get

$$\sigma^2_{P2} = \sigma_2^2 \; \frac{G^2}{K\omega^2} \; \frac{2\beta_2(K+\beta_2)^2 + K\omega^2}{(\beta_2^{\,2} + K\beta_2 + \omega^2)^2} \; . \tag{5.25}$$

Finally, use of the third-order Gauss-Markov model with covariance function (4.17) results in

$$\sigma^2_{P3} = \sigma_3^2 \; \frac{G^2}{K\omega^2} \; \frac{\omega^2(7K\beta_3^{\,2} + 3\beta_3 K^2 + K\omega^2 + 5\beta_3^{\,3}) + \beta_3^{\,2}(\beta_3 + K)^3}{(\beta_3^{\,2} + K\beta_3 + \omega^2)^3} \tag{5.26}$$

Equations (5.24) to (5.26) show that the elimination of time from the error function results in a considerable simplification of the expressions. When comparing the effect of different gravity field approximations we are primarily interested in the average behaviour of the error over long time intervals. Thus, equations (5.24) to (5.26) will serve our purpose well and will therefore be used in the next section.

The discussion so far has been restricted to the two horizontal channels, i.e. only the position errors due to unmodelled deflections of the vertical have been considered. Formulas for the height error due to unmodelled gravity anomalies or geoidal undulations can be derived in an analogous way. They can for instance be found in Jordan (1973). Due to the fact that the natural frequency of the vertical channel is higher, these errors are much smaller, usually less than 5% of those generated in the horizontal channels. They are therefore not given here.

The same procedure which was used to derive the steady state position error can be employed to determine the steady state velocity error induced by insufficient deflection information. In this case, the first diagonal term of the P_{SS}-matrix contains all the necessary information. Formulas can be found in Levine and Gelb (1969) and Jordan (1973).

6. RESULTS

To use the formulas developed in the last section, three steps are necessary. First, the different gravity field approximations have to be specified using the geodetic model described in section 4. Second, the essential parameters of the resulting covariance functions have to be determined. Finally, a suitable navigational covariance model must be selected by fitting the essential parameters of the geodetic model to those of the navigational model. After that, formulas (5.24) to (5.26) can be used to determine the position error induced by insufficient knowledge of the deflections of the vertical.

Gravity field approximations of increasing accuracy can be represented by spherical harmonic expansions of increasing degree and order N. When using such models in an airborne inertial navigator, the long wave-length features of the gravity field are eliminated. This means that with increasing N the unmodelled effects become more and more local in character. The covariance representation of geopotential models of arbitrary degree and order N can be simulated by subtracting the first N coefficients from the model (4.1). Covariance computations with such models are done in exactly the same way as with the full model. We thus have a consistent and simple way to characterize different gravity field approximations.

Table 6.1 lists the models used in the following computations and indicates reasons for their choice. The first three models represent gravity field approximations available today. The next three will hopefully become available on a global scale in the not too distant future. The last two represent models of well surveyed areas but cannot be considered as given globally. Since gravity field information is often given in gridded form, the grid interval corresponding to a geopotential model of degree and order N is also listed in the last column. It has been computed by the approximate formula

$$\text{grid interval} \approx \pi/N ,$$

which is useful for discrete function values but has limitations when applied to mean values, see e.g. Rapp (1977).

Degree and order N of geopotential model	Significance	gridded equivalent
N = 2	Normal gravity model	
N = 18	Geopotential model derived from orbital analysis of satellites.	10°
N = 36	Geopotential model from a combination of orbital analysis of satellites and terrestrial gravimetry.	5°
N = 90	Geopotential model from a combination of satellite altimetry and terrestrial gravimetry.	2°
N = 180	Geopotential model from a combination of terrestrial gravimetry and satellite to satellite tracking or satellite gradiometry.	1°
N = 300	as N = 180, using more optimistic assumptions.	36'
N = 720 N = 1080	Local gravity field representations obtained from terrestrial gravimetry or airborne gradiometry.	15' 10'

Table 6.1 : Gravity Field Approximations Used

The gravity field models listed in table (6.1) are needed at flying altitudes. As a typical range, flying heights between 10 km and 20 km have been chosen. The upward continuation of the covariance model (4.1) is obtained by an appropriate change of the radial distances r_P and r_Q. Again, the covariance computation is performed by the same algorithm as before.

The basic covariance model used to compute the models in table 6.1 has the p .ameters

$$A = 607.57 \text{ mgal}^2$$

$$k_n = \frac{1}{(n-1)(n-2)(n+24)} \qquad (6.1)$$

$$s = 0.998444$$

where

$$s = \left(\frac{R}{R_M}\right)^2$$

and $R_M = r_P = r_Q = 6371$ km

is the mean radius of the earth. As usual, the parameter A corresponds to the gravity anomaly covariance function. Its essential parameters are

$$C_o = 1762 \text{ mgal}^2$$

$$\zeta = 48\overset{'}{.}4$$

$$\chi = 13.7$$

They correspond somewhat better to the observational data than the parameters given by Tscherning and Rapp (1974). For a discussion, see Schwarz and Lachapelle (1980).

The different models in table 6.1 were computed using the subroutines COVAX by Tscherning (1976) and COVAPP by Sünkel (1979). A graphic representation of the results is given in figures 6.1 to 6.4. They show the covariance functions for the two deflection components at altitudes h = 10 km and h = 20 km. The gravity field approximation is characterized by the degree and order N of the geopotential model. The covariance functions N > 180 for h = 10 km and N > 90 for h = 20 km are too small to be represented in the given scale.

Three distinct features can be observed from these graphs. First of all, the covariance function of the η-component is much smoother than that of the ξ-component (note that the scales of the

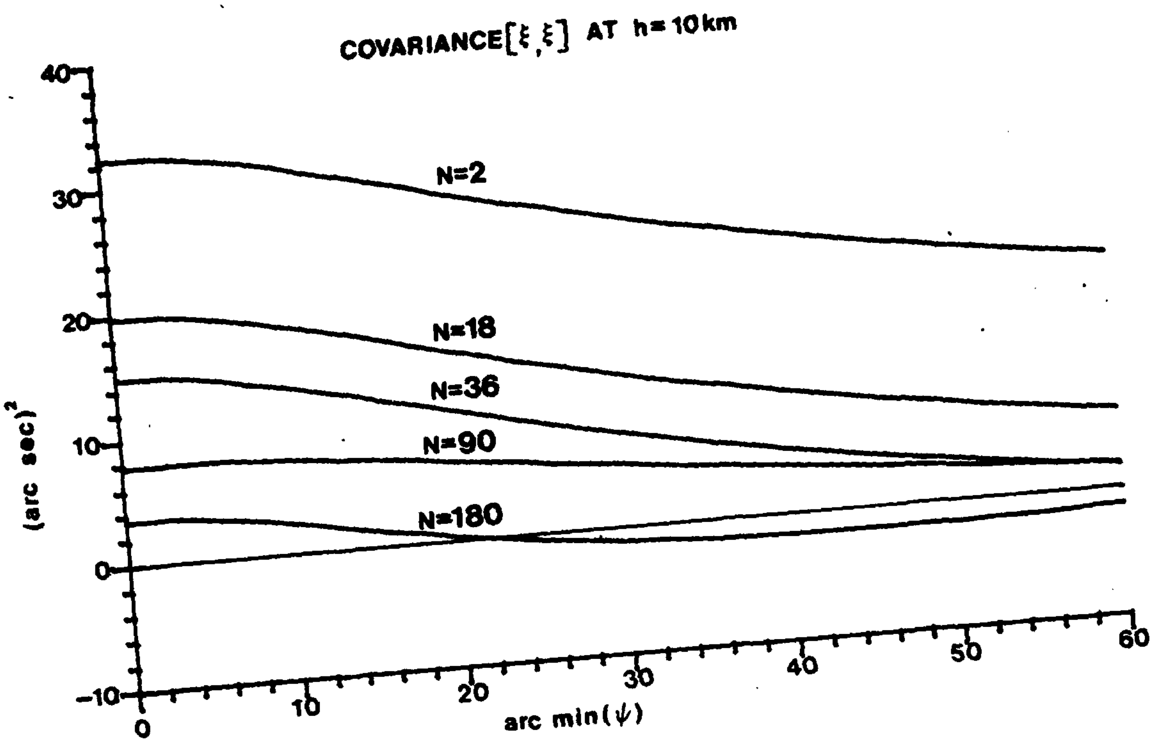

Fig. 6.1 Deflection covariance function C(ξ,ξ) at h = 10 km

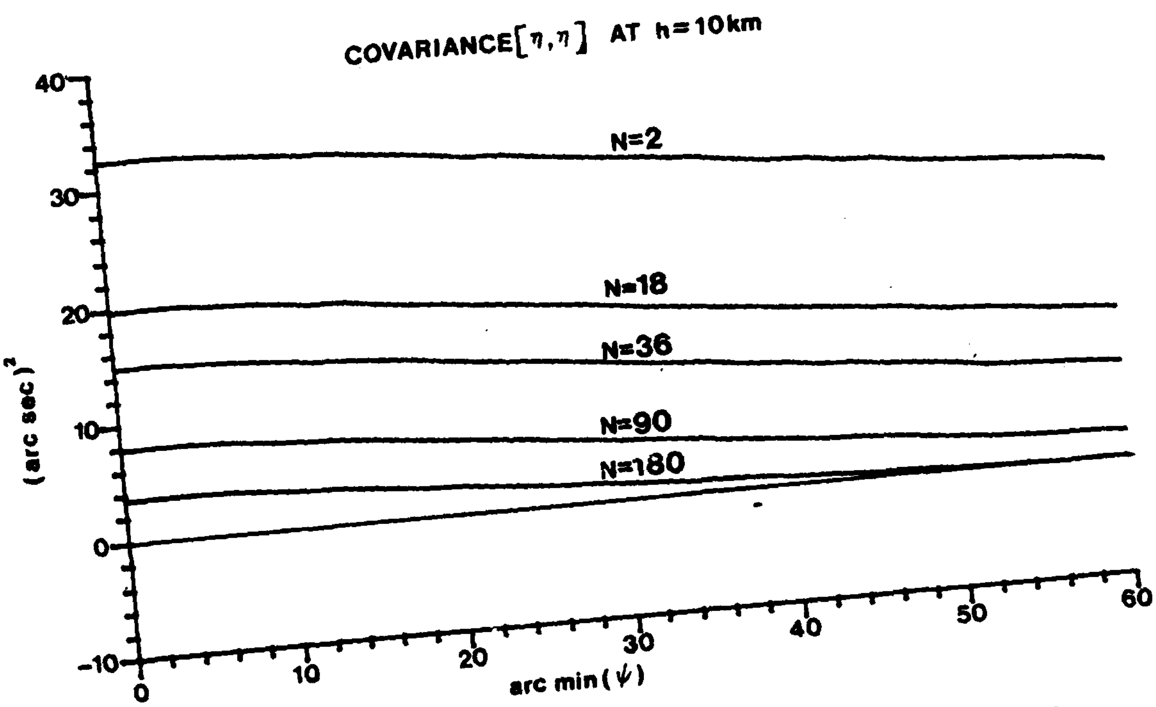

Fig. 6.2 Deflection covariance function C(η,η) at h = 10 km

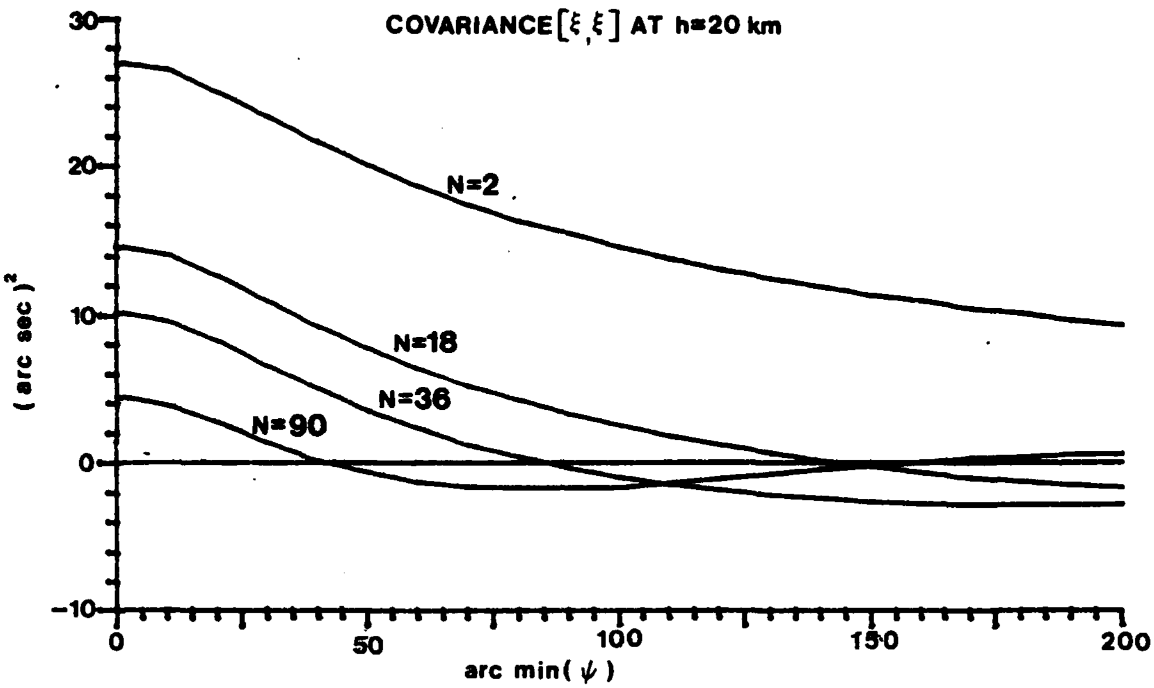

Fig. 6.3 Deflection covariance function C(ξ,ξ) at h = 20 km

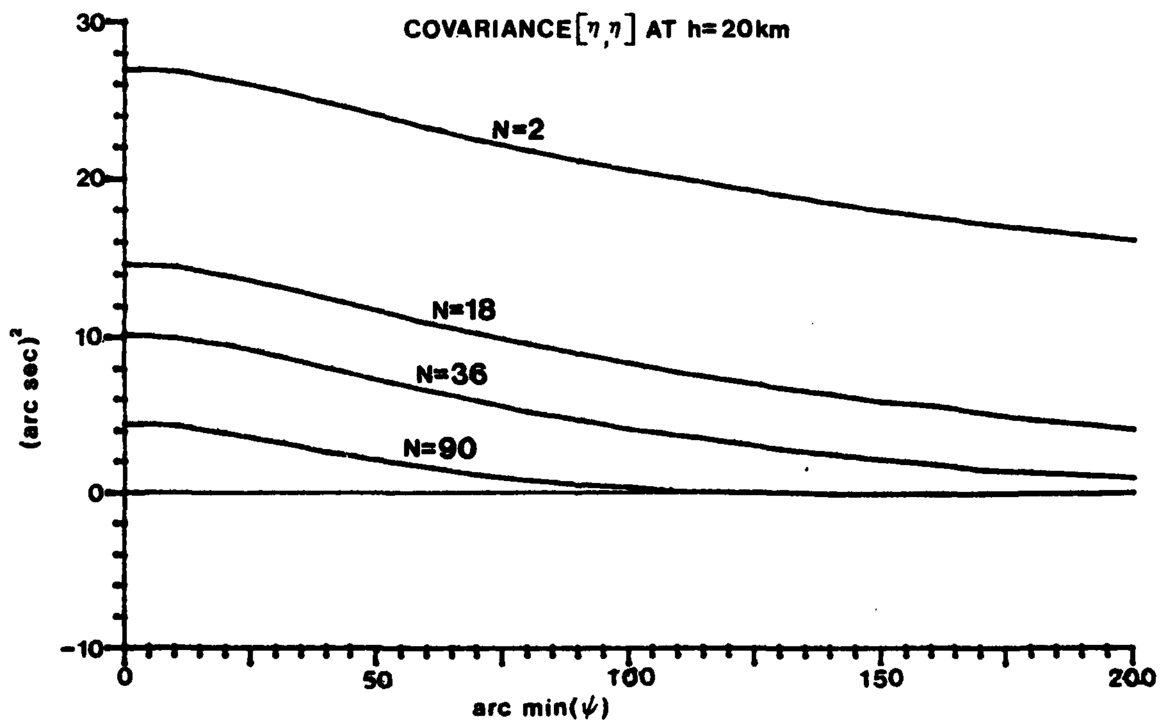

Fig. 6.4 Deflection covariance function C(η,η) at h = 20 km

ψ-axis are different between the two sets of figures). This means that we can expect a major difference in the correlation length. Second, the effect of increasing the degree and order N of the approximation is a reduction of both the variance and the correlation length of the covariance function. This feature is easily explained because the gravity field represented by the covariance function becomes more and more local with increasing N. Finally, the effect of upward continuation is a decrease in the variance but an increase in the correlation length. Again, this is physically meaningful because the high frequency spectrum is more affected by the attenuation with height. It should be noted that these results cannot be generalized to other functionals of the anomalous potential, as e.g. the geoidal undulations or second-order gradients. Since they are influenced by different parts of the spectrum their characteristical behaviour with respect to changes in N and h is quite different. For a discussion of this point, see Schwarz (1981).

The second step to the use of equations (5.24) to (5.26) is the determination of the essential parameters for all covariance functions. They can be obtained by applying the formulas of section 4 to the output of the subroutines mentioned above. Results are summarized in tables 6.2 to 6.5 which correspond to figures 6.1 to 6.4.

Besides the features already mentioned, the change in the curvature parameters χ is of specific interest. It decreases with increasing N and increasing h. Since the decrease with N is especially pronounced for the first few degrees, all models with $N \geq 18$ are quite well represented by covariance functions of second and third order Gauss-Markov models. This is an important result because it secures a good fit of the geodetic to the navigational models. It should be noted, however, that this is not true any more for inertial navigation on the surface of the earth. Due to the increase of χ with decreasing h, the curvature parameters tend to be too large for low-order Gauss-Markov models. That this is not a feature of the specific geodetic covariance model used can be gathered from a table of empirically

Model N =	Variance (arc sec)^2	Correlation length ζ (arc min)	distance $\bar{\zeta}$ (arc min)	Curvature parameter χ
2	32.5	81.2	136.	7.41
18	19.8	40.3	54.2	3.94
36	14.9	30.5	38.9	3.00
90	7.94	19.3	23.3	1.87
180	3.57	12.7	14.9	1.48
300	1.47	9.03	10.5	1.32
720	0.106	4.61	5.29	1.20
1080	0.0137	3.27	3.75	1.23

Table 6.2 : Essential Parameters of Deflection Covariance Function $C(\xi,\xi)$ for h = 10 km

Model N =	Variance (arc sec)^2	Correlation length ζ (arc min)	distance $\bar{\zeta}$ (arc min)	Curvature parameter χ
2	32.5	211.	387.	7.41
18	19.8	92.0	132.	3.94
36	14.9	66.4	89.9	3.00
90	7.94	39.2	49.7	1.87
180	3.57	24.7	30.2	1.48
300	1.47	17.1	20.5	1.32
720	0.106	8.51	10.0	1.20
1080	0.0137	6.00	7.05	1.23

Table 6.3 : Essential Parameters of Deflection Covariance Function $C(\eta,\eta)$ for h = 10 km

Model N =	Variance (arc sec)^2	Correlation length ζ (arc min)	distance $\bar{\zeta}$ (arc min)	Curvature parameter χ
2	27.0	113.	182.	6.50
18	14.6	52.5	67.8	3.02
36	10.1	38.8	47.9	2.22
90	4.35	23.5	27.8	1.57
180	1.42	15.0	17.3	1.32
300	0.390	10.3	11.8	1.24
720	0.00714	4.94	5.69	1.17
1080	0.000300	3.50	4.00	1.16

Table 6.4 : Essential Parameters of Deflection Covariance Function $C(\xi,\xi)$ for h = 20 km

Model N =	Variance (arc sec)^2	Correlation length ζ (arc min)	distance $\bar{\zeta}$ (arc min)	Curvature parameter χ
2	27.0	290.	511.	6.50
18	14.6	116.	159.	3.02
36	10.1	81.3	106.	2.22
90	4.35	46.2	57.1	1.57
180	1.42	28.3	34.0	1.32
300	0.390	19.1	22.6	1.24
720	0.00714	9.14	10.7	1.17
1080	0.000300	6.30	7.50	1.16

Table 6.5 : Essential Parameters of Deflection Covariance Function $C(\eta,\eta)$ for h = 20 km

determined χ-values published in Schwarz and Lachapelle (1980). They have an average size of $\chi = 9$ when referenced to a geopotential model of degree and order 22. This means that at $h = 0$ the simple navigational models, which typically have χ-values below 3, represent only local disturbances of the anomalous gravity field well. They will therefore give reliable estimates only when used with a gravity field approximation of high degree and order. Since, at present, geopotential models with a large enough N are not available, navigational models are needed which admit a larger curvature parameter. It is also interesting to note that for the same reason the covariance function of the first-order Gauss-Markov process is not a good model of the anomalous gravity field anywhere in the range $0 \leq h \leq 20$ km.

The third step to the use of equations (5.24) to (5.26) is the fitting of the geodetic models to one of the navigational models by way of the essential parameters. This amounts to obtaining a good fit for the curvature parameter because the variance and the correlation length of the two models can always be made to agree. Using the geodetic model parameters given in tables (6.2) to (6.5) a second or third order Gauss-Markov process is chosen depending on whether the χ-parameter in formula (4.16) or (4.19) is closer. Then, transforming $\bar{\zeta}$ into the time domain by

$$\beta_i = \frac{v}{\bar{\zeta}_i} \tag{6.2}$$

where v is the aircraft velocity, formula (5.25) or (5.26) can directly be applied. For a definition of terms in these equations, see formulas (5.3) to (5.5). A damping factor of 0.3 has been used throughout.

Results for aircraft velocities of 300 knots, 500 knots, and 800 knots are presented in tables 6.6 and 6.7. They show clearly that the single most influential factor for a reduction of position errors is the degree of the gravity field approximation model. Flying altitude, aircraft velocity, and correlation distance have some effect but are not decisive. Using one of the available geopotential models of degree and order 36 instead of the normal gravity field would reduce the errors by about 40% on average, and by 50% in case of a high speed

Model N =	C(ξ,ξ) 300 km (m)	C(ξ,ξ) 500 km (m)	C(ξ,ξ) 800 km (m)	C(η,η) 300 km (m)	C(η,η) 500 km (m)	C(η,η) 800 km (m)	C(ξ,ξ) and C(η,η) 300 km (m)	C(ξ,ξ) and C(η,η) 500 km (m)	C(ξ,ξ) and C(η,η) 800 km (m)
	Steady-state position error σ_p induced by								
2	197	208	207	180	186	195	267	279	284
18	162	149	128	154	162	161	224	220	206
36	135	117	98	140	141	131	194	183	131
90	69	53	41	97	78	61	119	94	73
180	37	28	22	53	41	32	65	50	39
300	19	15	12	28	21	17	34	23	21
720	4	3	3	5	4	3	6	5	4
1080	2	1	1	2	2	1	3	2	1

Table 6.6 : Gravity Induced Position Error for h = 10 km

Model N =	C(ξ,ξ) 300 km (m)	C(ξ,ξ) 500 km (m)	C(ξ,ξ) 800 km (m)	C(η,η) 300 km (m)	C(η,η) 500 km (m)	C(η,η) 800 km (m)	C(ξ,ξ) and C(η,η) 300 km (m)	C(ξ,ξ) and C(η,η) 500 km (m)	C(ξ,ξ) and C(η,η) 800 km (m)
	Steady-state position error σ_p induced by								
2	174	184	191	163	166	173	238	248	258
18	140	134	119	130	137	140	191	192	284
36	108	87	68	119	116	101	161	145	122
90	56	43	33	74	62	49	93	75	59
180	25	19	15	35	27	21	43	33	26
300	14	8	6	15	11	9	21	14	11
720	1	1	1	1	1	1	1	1	1
1080	0.2	0.2	0.1	0.3	0.2	0.2	0.4	0.3	0.1

Table 6.7 : Gravity Induced Position Error for h = 20 km

aircraft. With gravity models expected to become available in the near future (N = 180), the position error could be reduced to about 30 m to 50 m, i.e. to 15% or 20% of what it is today. From there, progress will be slow. To reach the meter range, a gravity field approximation equivalent to an expansion of degree and order 1000 would be necessary. This result is not surprising. It demonstrates the well-known fact that the medium and high frequency spectrum contributes considerably to the deflections of the vertical. Or, in other words, the relative contribution of local effects is not negligible.

Aircraft speed affects the position error to some extent. For all but the normal gravity field approximation a high aircraft velocity will improve results. The reason for this can be found in the dependance of the position error on β. The error curve increases to a β - value of about $3h^{-1}$ or $4h^{-1}$ and then falls off slowly, see e.g. figure 4 in Jordan (1973). Due to the long correlation distances of the normal model, a β - value of $3.5h^{-1}$, and thus the largest position error, is reached at about v = 640 knots for ξ and v = 1790 knots for η. For all the other models a speed of 300 knots corresponds to a β - value well above $3.5h^{-1}$. Thus, an increase in v will result in a decrease of the position error. It also explains why a large correlation distance reduces the error in the normal model and increases it in all the other models. An increase in flying altitude will in general reduce the position error. In this case, the smaller variance of the covariance curve more than compensates for the increase of the position error due to the larger correlation distance.

The position errors listed in tables 6.6 and 6.7 correspond to an average global behaviour of the anomalous gravity field. In areas where considerable deviations from this average behaviour occur, larger or smaller errors can be expected. An indication of the possible range of values is given in table 1 of Bernstein and Hess (1976). Although results are not directly comparable because mean gravity values and a somewhat different technique were used there, their 'best' and 'worst' case is quite close to what can be expected when using the empirical covariance functions in Schwarz and Lachapelle (1980).

The results also permit some conclusions on the effect of measuring errors in the data. Random errors will be strongly attenuated with height and will have a negligible effect on position accuracy. However, data errors which generate spatial correlations in the gravity field model will result in position errors of considerable size. Errors of this type are e.g. data biases over an extended area or blunders in discrete points which by way of upward continuation cause correlations. To give an indication of the possible effects, an error covariance function with a variance of (1 arc sec)^2 and different correlation distances has been used to compute the position errors in table 6.8. Although they are at present smaller

Variance (arc sec)^2	Correlation distance $\bar{\zeta}$ (arc min)	Position error 300 knots (m)	500 knots (m)	800 knots (m)
1	20	29	23	19
1	50	36	33	28
1	100	36	37	35
1	200	33	35	37

Table 6.8 : Position Error Due to Data Bias

than the effects coming from an insufficient gravity field approximation, they are large enough to warrant a careful analysis of the existing gravity field data to detect errors of this kind.

7. CONCLUSIONS AND RECOMMENDATIONS

Simplified analytical error models can be used with advantage to study the effect of gravity field approximations on the position accuracy of airborne inertial navigation. In this approach the components of the gravity vector have to be expressed as elements of a state vector which means that they are usually modelled as time correlated noise processes. Most of the processes proposed for this purpose in the navigational literature are of the Gauss-Markov type and imply a certain covariance structure of the anomalous gravity field. Since global geodetic covariance models cannot be cast into a form suitable for the state space approach, the question arises how well the navigational models approximate the geodetic covariance models. A direct comparison of the functions is not possible. However, using the essential parameter approach proposed by Mortiz, a meaningful comparison can be made. Results presented in this report show that none of the proposed navigational models gives a satisfactory fit when the normal gravity field is used as the basic approximation. Replacing the normal field by one of the available geopotential models ($18 \leq N \leq 36$) results in a good agreement between the geodetic covariance function at flying altitudes and the covariance function implied by a third-order Gauss-Markov process. For higher order reference fields second-order Markov processes give a better fit. The first-order Gauss-Markov process, often used in applications because of its simplicity, is not a good model for the anomalous gravity field. At the surface of the earth, none of the navigational models give a satisfactory agreement with all three essential parameters determined from actual data. Thus, to use these techniques in the marine and land-based environment, state space models are needed which give a better fit to empirical covariance functions.

Use of the analytical models show that the position error is mainly due to poorly modelled deflection components. The gravity anomaly and the geoidal undulations have a much smaller effect and can in general be neglected. The size of the error is strongly dependent on the degree and order of the gravity field model used. Flying altitude, aircraft velocity, and correlation distance have some effect but are not decisive. Using one of the available geopotential models of degree and order 36 instead of the normal gravity field will reduce the present position error of about $\sigma = 300$ m by 40% on average and by 50% in case of a high speed aircraft. With gravity models expected to become available in the near future ($N = 180$), the position error could be reduced to about 15% of its present size. From there on, progress will be slow. An approximation equivalent to an expansion of degree and order 1000 will be needed to reach the meter range.

The implementation of high-order reference fields in real time poses considerable problems of which the adequate representation of the available information is the most pressing one. Obviously, only interpolation methods with local support at flying altitude are feasible for this purpose. They will require storage capacity but will not be demanding in terms of computing power. Investigations on suitable representations of the gravity vector at flying altitude and on simple interpolation methods which can be executed in real time are needed to make practical use of these results.

Finally, errors in the gravity field data which generate spatial correlations in the model will result in position errors of considerable size. Typical errors of this type are biases between different data bases and every effort should be made to detect and eliminate them. Similarly, blunders in discrete points will generate correlations at flying altitude. In this case, the method used for upward continuation has some effect on the extent of the correlation and methods which reduce the influence of such errors should be developed.

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